Stable, accelerating universes in modified-gravity theories

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Modifications to gravity that add additional functions of the Ricci curvature to the Einstein-Hilbert action – collectively known as $f(R)$ theories – have been studied in great detail. When considered as complete theories of gravity they can generate non-perturbative deviations from the general relativistic predictions in the solar system, and the simplest models show instabilites on cosmological scales. Here we show that it is possible to treat $f(R) = R \pm \mu^4/R$ gravity in a perturbative fashion such that it shows no instabilities on cosmological scales and, in the solar system, is consistent with measurements of the PPN parameters. We show that such a theory produces a spatially flat, accelerating universe, even in the absence of dark energy and when the matter density is too small to close the universe in the general relativistic case.

Since it was first proposed as an explanation for the present period of cosmological acceleration [1], the idea of modifying the Einstein-Hilbert action of General Relativity with new functions of the Ricci curvature R has been the subject of a great deal of attention. In particular, its effect on the homogeneous cosmological expansion, on the growth and evolution of perturbations in a cosmological scenario, and on precision tests in the solar system, have all been investigated [2, and refs. therein].

To date, all of these investigations acknowledge that $f(R)$ gravity, considered as a complete theory, faces some serious problems arising from the set of extra degrees of freedom. For example, the addition of an $1/R$ term to the Einstein-Hilbert action is inconsistent with the first post-Newtonian correction to the metric outside a star [3], and also generates unstable solutions for spherically symmetric stars [4]. In the originally suggested formulation, it also leads to serious instabilities in the evolution of a homogeneous universe [5] and in the growth of perturbations in the early universe [6]. It is worthwhile to emphasize that all these instabilities are not related to Ostrogradski's theorem [7], but are nevertheless directly associated with the presence of higher derivatives in the Lagrangian of the gravitational field.

One avenue that can suppress the instabilities of $1/R$ gravity is to derive the field equations from from the Langrangian in the so-called Palatini formalism, *i.e.*, by assuming that the connection and the metric are independent fields [8]. This reduces the order of the field equations and resolves the instabilities.

However, it was recently shown that the field equations that are generated with the Palatini formalism do not allow for consistent solutions of the metrics of polytropic stars [9]. As a result, this approach cannot be considered viable for gravity theories with non-linear Lagrangian actions. While there are many, more elaborate, functions of R that may produce viable theories [10, 11], $1/R$ gravity appears in general to suffer from the pathological degrees of freedom.

Matters change, however, if we consider a polynomial $f(R)$ Lagrangian to be a truncation of a perturbative expansion of a more general theory. This, as we shall see, can radically change the properties of the system. There are two general ways that can lead to a perturbative expansion in power of the curvature.

In one case, the $f(R)$ gravity can be simply a Taylor (or Laurent) expansion of a non-polynomial Lagrangian. It is often the case that the naive pathologies of the field theory do not appear in the more general actions. there exists a large class of theories whose perturbative approximations produce additional, fictional degrees of freedom when treated naively – including not only the semi-classical limit of Quantum Electrodynamics, but also the computation of String Theory gravitational corrections to General Relativity [12].

In these cases, while the fundamental theory is second-order and local, the effective field theory is higher-order and will generically lead to unstable, and even fictional, degrees of freedom even in regimes where the perturbative quantities are small [13].

In a different situation, $f(R)$ gravity is an expansion of a fundamental theory that is of second-order but non-local. In this case, the expansion generically introduces fictional degrees of freedom even in regimes where the perturbative quantities are small. Such pathologies, however, can be removed in the same, mathematically rigorous, and consistent fashion to restore the correct behavior of the theory [14].

In the end, however, the origin of the justification does not matter for the actual results of the analysis which are the same in each case; we may simply consider $1/R$ gravity to be a perturbative approximation ("perturbative $f(R)$") instead of considering it an exact theory ("exact $f(R)$.") This fundamentally changes the analysis of various cosmological and astrophysical phenomena. This interpretation removes the additional degrees of freedom from the field equations and, as we show below, it cures the theory from classical instabilities. In addition, the perturbative and solar system properties are radically changed.

It is important to emphasize here that while our approach shares some (but not all) of the mathematical methods associated with effective field theories in the quantum realm, the reasoning behind the constraints imposed upon the apparent degrees of freedom is not as restricted.

In effective field theories, there is a high-energy ("ultraviolet") limit above which we expect new physics to appear – allowing us to freeze out degrees of freedom that have mass scales above this limit because conservation of energy limits their excitation.

Here, by contrast, the mechanism that prevents the excitation is left unspecified; the techniques of the analysis allow us to produce a consistent answer in the perturbative regime. Such techniques can be applied not only to constraints from energy conservation but also, for example, to non-local theories where the constraint is implicit, or to cases where non-canonical kinetic terms constrain perturbative degrees of freedom to track a cosmological background [15]. We need assume only that such a constraint exists, and how it appears to leading order in some limit, to ask its perturbative consequences.

In the first section, we introduce the main techniques of our analysis, and demonstrate that, even within our perturbative approach, the $1/R$ term in the action leads to late-time cosmic acceleration. We discuss the limitations of the approach for providing definite predictions of the homogeneous expansion.

In the second section we examine the behavior of cosmological perturbations to demonstrate that there are no instabilities for the theory in the matter dominated regime. In the third section we show that perturbative $f(R)$ gravity can pass solar system tests at the current level of experimental sensitivity. Finally, we summarize our results and discuss the provocative ways they reflect upon studies of $f(R)$ gravity.

I. HOMOGENEOUS EXPANSION TO FIRST ORDER.

A great variety of functions $f(R)$ for the Lagrangian of the gravitational field have been proposed. The original suggestion, $f(R)$ taken to be $R - \mu^4/R$ (the "CDTT" case), suffers a number of problems under the exact paradigm. However, as we show here, under the perturbative analysis it survives very well, and we shall consider it in detail.

Assuming an expanding universe with signature $(-,+,+,+)$ and spatial curvature k, we first compute the "exact" equation of motion for $R+\mu^4/R$ gravity (for convenience, we flip the sign – CDTT can be considered to have a negative value for μ^4.) We find, in accordance with the literature,

$$\left(1-\frac{\mu^4}{R^2}\right)R_{\mu\nu}-\frac{1}{2}\left(1+\frac{\mu^4}{R^2}\right)Rg_{\mu\nu} -\mu^4[g_{\mu\nu}\nabla_\alpha\nabla^\alpha-\nabla_{(\mu}\nabla_{\nu)}]R^{-2} \;=\; 8\pi G T_{\mu\nu}\;.$$

There are two independent components in the Einstein equation; we can take the time-time and trace components. Respectively,

$$\frac{R}{2}-\frac{3\ddot{a}}{a}+\mu^4\left(\frac{1}{2R}+\frac{6\dot{a}\dot{R}}{aR^3}+\frac{3\ddot{a}}{aR^2}\right) \;=\; 8\pi G\rho, \quad (1)$$

$$R+\mu^4\left(\frac{3}{R}+\frac{18\dot{a}\dot{R}}{aR^3}-\frac{18\dot{R}^2}{R^4}+\frac{6\ddot{R}}{R^3}\right) \;=\; -8\pi G\rho, \quad (2)$$

where R, the Ricci scalar, is

$$R=\frac{6(\dot{a}^2+a\ddot{a}+k)}{a^2}, \quad (3)$$

and overdots are derivatives with respect to time.

Whatever the matter-energy content of the Universe, since R contains second derivatives of a, this set of equations is fourth order and there are thus two extra degrees of freedom. Here we have, for simplicity, assumed that all matter is pressureless dust – reasonable for large scales in the "matter dominated" regime.

In most work on $f(R)$ gravity, the equations are rewritten so that these extra degrees are absorbed into a scalar ϕ field that is governed by a second order equation of motion. We wish, on the other hand, to consider μ^4/R as only the first term in a series expansion. We remain agnostic about what the next term looks like, as indeed we are allowed to do in the context of a perturbative expansion.

We will thus seek a solution to Eqs. 1 and 2 valid only to $\mathcal{O}(\mu^4)$. To do so is simple. For the terms above multiplied by μ^4, we insert the zeroth order solutions – *i.e.*, the solutions to the ordinary Friedman equation.

Doing so, and using the fact that the stress-energy tensor is conserved even in $f(R)$ gravity [16], we find the new, modified Friedman equation to be

$$H^2=\frac{8\pi G\rho}{3}-\frac{k}{a^2}-\frac{3\mu^4}{32\pi^2G^2\rho^2}\left(\frac{k}{a^2}-\frac{8\pi G\rho}{3}\right)+\cdots, \quad (4)$$

where the dots remind us that this approximation is good only to first order.

From this result, we see that a standard matter-dominated era exists in a perturbative $1/R$ theory; this is in direct contrast to the behavior in exact $1/R$ [2, 17, 18], where the usual $t^{2/3}$ scaling of the matter era is unstable to a so-called "ϕMDM era", where the scaling is now $t^{1/2}$. In both theories the matter dominated solution exists; the suppression of the extra degrees of freedom makes it stable in perturbative $1/R$.

By inspection of Eq. 4, one can see that in perturbative $1/R$ it is possible to have a spatially flat universe even if the matter density appears insufficient in the General Relativistic case. Further, one can see that many choices for k and μ^4 will lead to accelerated expansion. Fig. 1 shows the various regimes, for different values of μ^4 and the matter density, the latter phrased as the "classical" GR quantity $8\pi G\rho/3H^2$, equal to Ω_m in the μ equal to

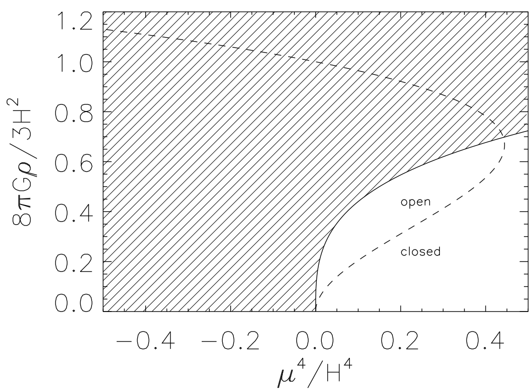

FIG. 1: The phase space of perturbative $1/R$ gravity. The value of μ, referenced to the Hubble parameter, is on the horizontal axis and the matter density is on the vertical axis. The unshaded region is where one finds acceleration, and the dashed line is the locus of points for which the spatial curvature is zero. This graph shows that when $(\mu/H)^4 \simeq -0.2$, cosmic acceleration *and* a spatially flat universe are consistent with the amount of matter in the universe inferred by traditional methods.

zero case. Indeed, for $(\mu/H)^4 \simeq 0.2$, a classical GR value of $\Omega_\mathrm{m} \simeq 0.3$ leads to both a spatially flat universe and to late-time cosmic acceleration. It is important to recognize that if the "control parameter" of the perturbative expansion is taken to be $(\mu/H)^4$, this solution approaches the limit in which one expects results to be reliable; the extent to which the onset of "lambda domination" allows for perturbative approaches requires additional study.

Acceleration in our model begins to appear at high curvatures where the perturbative expansion is valid. However, interpreting this plot requires a degree of caution: in particular, it must be emphasized that the results here are good only to $\mathcal{O}(\mu^4)$; whether or not we can reproduce, e.g., the currently measured deceleration parameter at refshift zero, would require knowledge of the full theory of which $1/R$ is only an approximation.

When the relevant physical quantities – either the curvature squared, R^2, or the matter density, $(G\rho)^2$ – become of order μ^4, the perturbative analysis here can no longer be trusted and the complete field equation needs to be solved to generate predictions. By construction, the present day universe is in a regime where these assumptions break down and, hence, quantitative predictions of this theory for comparison with observations will require specification of the theory beyond the order considered here.

Note also that, as expected, the two extra degrees of freedom have disappeared; simply specifying the scale factor and its derivative at some early time is sufficient to determine the complete future evolution of the system. The phenomenon of a finite-term approximation to an infinite series in an action or equation of motion leading, in a naieve treatment, to extraneous degrees of freedom, is well known. Our manipulations above are a canonical solution to the problem [14].

II. COSMOLOGICAL PERTURBATIONS.

As is well known, the CDTT case, considered as an exact theory, is unstable to perturbations on cosmological scales. This can be seen by an examination of CDTT gravity in the Einstein frame where it is found that the extra degrees of freedom – considered now as a scalar field – show an instability on superhorizon scales that grows larger at earlier times as the curvature increases [19].

While the opposite sign, *i.e.*, the action $R + \mu^4/R$, is stable even in the exact case, and Fig. 1 shows that it is this choice that leads to acceleration at first order, it is worth noting that the stability of the evolution does not depend on the sign of μ^4 and instead is related to the disappearance of these extra degrees of freedom.

This can be seen, trivially, by examining the perturbative Friedman equation, Eq. 4. At early times, when the matter density ρ is much larger than μ^4, we recover the classical Einstein solution and perturbations in the curvature grow in the usual Einstein-de Sitter fashion. The growth occurs on timescales of order $1/H$ and not, as in the case of the "exact" theory, on the much shorter timescale of order μ^2/H^3.

The stability of our solution is related to the absence of additional degrees of freedom that would allow one to relax into a high-density but low-curvature regime. As we can see from Eq. 2, the deviation from the classical curvature-density relation is constrained to be of $\mathcal{O}(\mu^4)$ or smaller.

III. SOLAR SYSTEM TESTS.

We finally consider solar system constraints. In particular, we examine whether or not tight constraints on the Parametrized Post-Newtonian (PPN) parameters can rule out $1/R$ gravity considered as a perturbative expansion. As suggested by Ref. [20], and reasserted by Ref. [3], non-perturbative $1/R$ gravity is equivalent to a scalar-tensor theory that may have already been ruled out using solar system tests.

Our analysis of the problem will be conceptually similar to that of our cosmological investigation; we note that are results are in agreement with a "post-Newtonian" (*i.e.*, weak field) analysis under a similar "perturbative" interpretation of $1/R$ [21]. We will at all times make our approximations to $\mathcal{O}(\mu^4)$, and test the assumption that the solution stays perturbatively close to that found for General Relativity. Since our perturbative expansion can not handle the vacuum regime, we will assume that $(G\rho)^2$ always remains much larger than μ^4 – a very reasonable assumption given the ambient densities of the solar system.

The relevant equation of motion is the trace equation [3], written here without reference to a particular background spacetime as

$$\Box\frac{\mu^4}{R^2} - \frac{R}{3} - \frac{\mu^4}{R} = \frac{8\pi GT}{3}. \tag{5}$$

Inside the star – in the presence of matter – the background Ricci curvature that we perturb around is $8\pi G\rho$. We define $c(r)$ to be the (dimensionful) departure from the background solution:

$$c(r) \equiv R^2(r) - (8\pi G\rho)^2 \quad [= \mathcal{O}(\mu^4) \text{ or higher}]. \tag{6}$$

We then rewrite Eq. 5 in terms of $c(r)$ and $\rho(r)$, finding

$$\begin{aligned} &-3(8\pi G\rho)^2\mu^4\left[\frac{(8\pi G\rho)^5 r}{2} - rc'^2 - (16\pi G)^2 r\rho\rho' c'\right.\\ &\quad +\frac{1}{2}(8\pi G\rho)^2(2c' - 6(8\pi G)^2 r\rho'^2 + rc'')\\ &\quad \left.\cdot + (8\pi G)^4\rho^3(2\rho' + r\rho'')\right] - c\left(\frac{1}{2}(8\pi G\rho)^7 r\right.\\ &\quad -\mu^4\left[\frac{3}{4}(8\pi G\rho)^5 r - 9rc'^2 - 36(8\pi G)^2 r\rho c'\rho'\right.\\ &\quad 3(8\pi G\rho)^2(10(8\pi G)^2 r\rho'^2 - 2c' - rc'')\\ &\quad \left.\left.\cdot + 6(8\pi G)^4\rho^3(2\rho' + r\rho'')\right]\right) = 0, \end{aligned} \tag{7}$$

where primes denote derivatives with respect to radius. If we assume that the first and second derivatives of $c(r)$ are of order $c(r)/r$ and $c(r)/r^2$, we can solve Eq. 7 for $c(r)$ self-consistently to $\mathcal{O}(\mu^4)$. We find

$$c(r) = -\mu^4\left(6 + \frac{3\rho'}{G\pi r\rho^2} - \frac{9\rho'^2}{2G\pi\rho^3} + \frac{3\rho''}{2G\pi\rho^2}\right), \tag{8}$$

This expression is finite for physical boundary conditions at the origin. Note that the coefficient of μ^4 here may become large depending on the system; however, if μ^4 is chosen to be the cosmic acceleration scale, the dimensionless departure from General Relativity – *i.e.*, $c(r)/(G\rho)^2$ – is of order

$$\alpha = \frac{r_{\rm curv}^6}{r_{\rm object}^2 r_{\rm horizon}^4}, \tag{9}$$

where $r_{\rm horizon}$ is the Hubble distance, $r_{\rm object}$ is the size of the object, and $r_{\rm curv}$ its (gravitational) radius of curvature, $c/\sqrt{G\rho}$. For a laboratory experiment, α is 10^{-32}, for the Sun, 10^{-50}, for the inner solar system, 10^{-35}, and for the galaxy 10^{-13} – easily satisfying the current PPN bounds. In all these cases the assumption made in the course of the solution about the derivatives of $c(r)$ is consistent, as can be seen by examination.

We note that our solution here is conceptually distinct from the debates as to the viability of exact $f(R)$ gravity in the solar system. It has been suggested [2, 18] that a non-cosmological ambient density in the solar system might allow us, through a "chameleon" effect, to avoid the no-go analysis of Ref. [3]. In those analyses, the mass of the extra degrees of freedom becomes large enough to "freeze out" as in a standard effective field theory. Our analysis here does also rely on a non-zero ambient density, but for an entirely different reason – to satisfy the validity of a perturbative approximation to a more fundamental theory. We reach equivalent results in the solar system, but not on cosmological scales.

IV. CONCLUSIONS.

In this work, we have examined $f(R)$ gravity in a perturbative fashion and considered its effects in cosmological and solar-system observations. Instead of considering it an exact theory, whose higher-order derivatives demand the introduction of new degrees of freedom, we require it to be perturbatively close to GR. Such an approach removes the extra degrees of freedom and greatly simplifies the theory without making it trivially equivalent to GR.

We have demonstrated a number of facets of the theory. We have shown that it avoids instabilities that exist in $f(R)$ gravities on cosmological scales, and also that it can, to leading order, induce acceleration. Further we have shown that in the solar system the theory is consistent with current measurements of the PPN parameters.

The use of our work, at the intersection of observational cosmology and fundamental physics, is twofold.

First, we draw attention to the fact that many common modifications to gravity lead to radically different results if viewed as perturbative approximations to a more fundamental theory. In several cases, the approach resolves several traits of the theories that would be, otherwise, prohibitive.

Second, we provide an encouraging note for possible experimental verification of the presence of non-linear terms in the action of the gravitational field. Broadly speaking, modifications to gravity that look, in approximation, like the kinds of $f(R)$ expansions we address here, may be considered *prima facie* reasonable despite the mounting evidence against their being the exact deviations from the Einstein-Hilbert action.

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